SUMO

Scottish University Mathematics Olympia

Saturday 5 March 2022

Instructions and guidance:

- Do not turn over until told to do so.
- You will have 4 hours to solve as many questions as possible, each worth 10 points.
- Clearly write your team-name at the top of every piece of paper you wish to be marked.
- Use a black or blue pen, or a dark pencil. Rulers, compasses, protractors, rubbers and a non-programmable calculator may be used but will not be required.
- Devices with internet connectivity are strictly prohibited, and must not be used throughout the duration of the competition.
- You may ask any of the exam invigilators to provide definitions or clarifications for any of the questions.
- There is an appendix at the end of this booklet containing various definitions and examples.
- One complete solution will be awarded more points than several partial solutions.

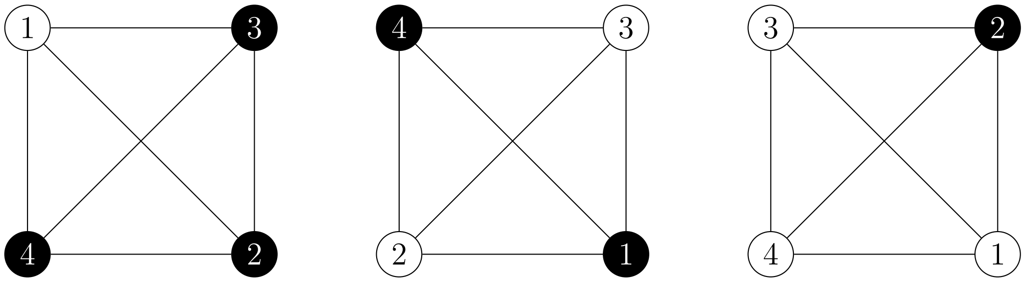
Prepared by the St Andrews University Mathematics Society,
with thanks to Peiran Wu and Hyeokjun Kwon.

1 Problems

Problem 1. Show that there exists a set of 100 consecutive natural numbers that contains exactly 10 primes.

Problem 2. Four different coins are placed on the corners of a square. Each coin is labelled with a unique number from $\{1, 2, 3, 4\}$ and has two faces, one black and one white. We define a *swap* on coins i and j as the process of flipping each coin (black becomes white, and white becomes black), and then interchanging the position of coin i and coin j on the square. For an example of some valid swaps, see Figure 2, in the appendix at the end of the document.

Two configurations of coins are said to be *equivalent* if and only if you can reach one from the other by applying a sequence of swaps. Which, if any, of the following positions are equivalent?



(a) (b) (c)

Problem 3. Let n be a natural number, and consider a connected graph G on n vertices. To each edge in G, we assign a real number, and denote this the *weight* of that edge. For each path in G, we define the *speed* of the path to be the largest weight of any edge within that path. For any two distinct vertices x, y in G, let $f(x, y)$ denote the smallest speed of any path connecting x with y. Prove that the cardinality of the range of f is not greater than $n - 1$.

Problem 4. Let G be a group with identity e and $\phi : G \to G$ a function such that

$$\phi(g_1)\phi(g_2)\phi(g_3) = \phi(h_1)\phi(h_2)\phi(h_3)$$

whenever $g_1 g_2 g_3 = e = h_1 h_2 h_3$. Prove that there exists an element $a \in G$ such that $\psi(x) = a\phi(x)$ is a homomorphism (i.e. $\psi(xy) = \psi(x)\psi(y)$ for all $x, y \in G$).

Problem 5. Ann and Bob play the following game. First they draw an $n \times n$ grid and colour its top and bottom sides red, and its left and right sides blue. Ann starts the game, and then they alternate taking turns. During this game, Ann will be playing with red, and Bob will be playing with blue. On Ann's turn, she chooses a square within the grid which is not yet coloured and which has at least one red side, and colours it red. Bob plays similarly: he chooses a square within the grid which is not yet coloured and which has at least one blue side, and colours it blue.

We define a *chain* to be a sequence of adjacent (not including diagonals) squares that are all the same colour. The game can end in one of three ways:

1. Ann connects the top and bottom sides with a chain of red squares, in which case she wins;
2. Bob connects the vertical sides with a chain of blue squares, in which case he wins;
3. the player whose turn it is cannot make any legal move, in which case the game is a draw.

Determine the values of n for which one of the two players has a winning strategy.

Problem 6. Let R be a subring of the ring of 4×4 matrices over the field $\mathbb{Z}/2\mathbb{Z}$ (the integers modulo 2). Note that R contains the 4×4 identity matrix. Prove that the set of units of R does not have cardinality 5.

Problem 7. Let A be a set. We say that A is *selfish* if the cardinality (number of elements in a set), $|A|$, is an element of A. Prove that the number of subsets of $\{1, 2, \ldots, n\}$ which are *minimal selfish* sets (that is, selfish sets none of whose proper subsets is selfish) is the n-th Fibonacci number.

Problem 8. An integer n, unknown to you, has been randomly chosen in the interval $[0, 2022]$ with uniform probability. Your objective is to select n in an **odd** number of guesses. After each incorrect guess, you are informed whether n is higher or lower than your previous guess, and you **must** guess an integer on your next turn among the numbers that have not yet been ruled out. Show that there is a strategy such that the chance of winning is greater than $2/3$.

Problem 9. Let $h \neq 0$ be a real number and D_h an operator that acts on the set of real functions, such that for any function $f : \mathbb{R} \to \mathbb{R}$,

$$D_h(f) = \frac{f(x+h) - f(x)}{h}.$$

Find all values of h for which there exists $f : \mathbb{R} \to \mathbb{R}$ such that

$$D_h(D_h(f)) = D_h^2(f) = f \text{ but } D_h(f) \neq f.$$

Problem 10. A haggis is a small fluffy creature that lives in the heart of the Scottish Highlands and, amongst haggis enthusiasts, it is a well known fact that they love running in circles, and they have very loose fitting socks. Henry the haggis is no exception! Henry is currently running in a loop around a hill. The loop is of length $60\sqrt{2}$ metres. Every 20 metres Henry runs, he pauses to pull his socks up. Show that given any point P on the loop and any $\varepsilon \in \mathbb{R}$ greater than 0, Henry will eventually pull his socks up within ε metres (measured along the loop) of the point P.

Problem 11. Consider a positive integer k and two other integers $b > w > 1$. There are two strings of pearls: a string of b black pearls and a string of w white pearls. The length of a string is the number of pearls on it.

A series of cuts are made on the strings of pearls in accordance with the following rules:

(i) The strings are ordered by their lengths in a decreasing order. If there are some strings of equal lengths, then the white ones precede the black ones. Starting with the largest, select the first k strings that contain more than one pearl; if there are fewer than k strings longer than 1, select all of them.

(ii) Next, cut each select string into two parts differing in length by at most one.(For instance, if there are strings of black pearls of length 5, 4, 4, 2; strings of white pearls of length 8, 4, 3; and $k = 4$, then the strings of 8 white, 5 black, 4 white and 4 black pearls are cut into the parts $(4, 4), (3, 2), (2, 2)$ and $(2, 2)$, respectively).

(iii) Repeat until the first isolated white pearl appears

Prove that, at the stage in which the cutting process terminates, there will exist a string of at least two black pearls.

Problem 12. Consider the sequence $a_1, a_2, \ldots$ defined as follows:

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 2 \\ a_{n+2} &= a_n + 2a_{n+1} \qquad \text{for all } n \in \mathbb{N} \setminus \{0\} \end{aligned}$$

Show that

(a) for all i, if a_i is prime then i is prime;

(b) for all $k \in \mathbb{N}$, there exists $d, m \in \mathbb{N} \setminus \{0\}$ such that $dk = a_m$; that is, for each integer k, we can find an a_m such that a_m is a multiple of k.

Problem 13. Define $a = 1 + \sqrt{2},\ b = 1 + \frac{1}{\sqrt{2}}$
A country has one infinitely long straight motorway, on which the government puts in mile markers, every mile, at $1, 2, 3 \ldots$.

Unfortunately the country has no mobile phone network. So the Car Club installs an emergency phone every a miles, starting at a. Then the Motor Group outdoes them by installing an emergency phone every b miles, starting at b.

Next, the government then decides that if any pair of consecutive mile markers has no emergency phone between them, the government will install an emergency phone there.

Where (if anywhere) would the first government emergency phone go?

Problem 14. Let G be a finite group and suppose the automorphism T sends more than three-quarters of the elements of G onto their inverses. Prove that $T(x) = x^{-1}$ for all $x \in G$ and that G is abelian.

Problem 15. Let n_0 be a four digit number that uses at least two different digits with leading zeros allowed. Consider the following process:

1. Rearrange the digits of n_i to form the largest possible four digit number, a_i, and the smallest possible four digit number, b_i, with leading zeros if necessary.

2. Set $n_{i+1} = a_i - b_i$.

3. Repeat.

Prove that, for any four digit starting number n_0 that uses at least two different digits, $n_k = 6174$ for all $k \geq 7$. (Please don't just brute force every eligible four digit number, we do have to mark your answers!)

2 Appendix

Problem 2. Examples of valid coin swaps:

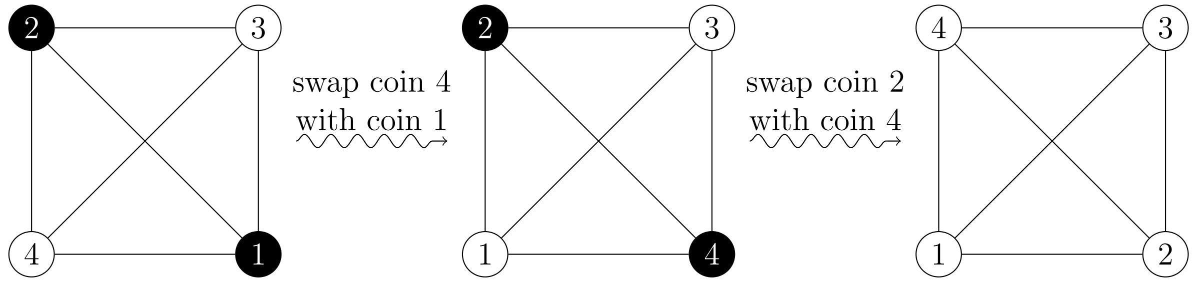


Figure 2: Possible coin swaps

Problem 3.

Definition 3.1 (Graph, Vertex, Edge)**.** A *graph* G is the ordered pair $G = (V, E)$ where:

- V be a set, refereed to as the set of *vertices*;
- $E \subseteq \{\{x, y\}, : x, y \in V \text{ and } x \neq y\}$, referred to as the set of *edges*.

Definition 3.2 (Walk, Path)**.** Let $G = (V, E)$ be a graph. A *walk* is a sequence of edges $e_1, e_2, \ldots, e_{n-1}$ for which there is a sequence of vertices $v_1, v_2, \ldots, v_n$ such that $e_i = \{v_i, v_{i+1}\}$ for all $i = 1, 2, \ldots, n-1$.

A *path* is a walk in which all vertices are distinct.

Definition 3.3 (Connected)**.** Let $G = (V, E)$ be a graph. We say that G is *connected* if for all a, b in V, there exists a path between a and b.

Problem 4.

Definition 4.1 (Group)**.** A *group* is a set G equipped with a binary operation " $\cdot$ " (where the result of the operation on $x, y \in G$ is $x \cdot y \in G$) such that all of following hold:

G1. for any $x, y, z \in G$, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$;

G2. there exists an element $e \in G$ such that, for any $x \in G$, $x \cdot e = e \cdot x = x$;

G3. for any $x \in G$, there exists $x^{-1} \in G$ such that $x \cdot x^{-1} = x^{-1} \cdot x = e$.

For brevity, we usually drop the " $\cdot$ ", and write xy to mean $x \cdot y$.

Problem 6.

Definition 6.1 (Ring)**.** A *ring* is a set R equipped with two binary operations " $+$ " and " $*$ " (where the results of the operations on $a, b \in R$ are $a + b \in R$ and $a * b \in R$ respectively) such that all of following hold:

A1. for any $a, b, c \in R$, $a + (b + c) = (a + b) + c$;

A2. there exists an element $0 \in R$ such that, for any $a \in R$, $a + 0 = 0 + a = a$;

A3. for any $a \in R$, there exists $-a \in R$ such that $a + (-a) = (-a) + a = 0$;

A4. for any $a, b \in R$, $a + b = b + a$;

M1. for any $a, b, c \in R$, $a * (b * c) = (a * b) * c$;

M2. there exists an element $1 \in R$ such that, for any $a \in R$, $a * 1 = 1 * a = a$;

D. for any $a, b, c \in R$, $(a + b) * c = a * c + b * c$, and $a * (b + c) = a * b + a * c$.

Definition 6.2 (Subring)**.** Let R be a ring. A *subring* of R is a subset S of R such that

(i) $1 \in S$;

(ii) if $x, y \in S$, then $x + y$, $x * y$, $-x \in S$.

Definition 6.3 (Unit)**.** Let R be a ring. A *unit* of R is an element $x \in R$ for which there exists $y \in R$ such that $x * y = y * x = 1$.

Problem 14. For the definition of a group, see Definition 4.1.

Definition 14.1 (Automorphism)**.** Let G be a group. An *automorphism* of G is a function $\phi : G \to G$ such that

(i) $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in G$;

(ii) ϕ is a bijection.