SUMO

Scottish University Mathematics Olympiad

Saturday 25 March 2023

Instructions and guidance:

- Do not turn over until told to do so.
- You will have 3 hours to solve as many questions as possible, each worth 10 points.
- Clearly write your team name at the top of every piece of paper you wish to be marked.
- Use a black or blue pen, or a dark pencil. Rulers, compasses, protractors, rubbers and a non-programmable calculator may be used but will not be required.
- Devices with internet connectivity are strictly prohibited, and must not be used.
- You may ask any of the exam invigilators to provide definitions or clarifications.
- There is an appendix at the end of this booklet containing definitions and clarifications.
- One complete solution will be awarded more points than several partial solutions.

Prepared by the mathematics societies of

University of Aberdeen
University of Glasgow
Heriot-Watt University
University of St Andrews

with thanks to

Jordan Baillie
Dr Collin Bleak
Prof Kenneth Falconer
Daniel Hinds
Rhys Murphy

1 Problems

Problem 1. Construct an expression of three variables $a, b, c \in \mathbb{R}$ that is equal to $\min(a, b, c)$ using only a finite number of the following operations:

- Addition $+$
- Subtraction $-$
- Multiplication $\times$
- Division $\div$
- Absolute Value $|\cdot|$

Problem 2. This problem has been edited out due to copyright reasons.

Problem 3. The SUMO problem setting committee play an elaborate game involving rock-paper-scissors. It begins with two members M_1 and M_2 sitting at a table, and another six $M_3, M_4, M_5, M_6, M_7, M_8$ standing in a queue waiting to play, with M_3 at the front.

After each rock-paper-scissors match, the winner gains one point and remains seated. The loser goes to the back of the queue and the first member in the queue sits opposite the winner. The game stops whenever a player reaches a score of 7 points. At the end of the game, the committee notice that a total of 37 rock-paper-scissors matches were played. Which member won and why?

Problem 4. Let A, B, C, D be points in the plane such that $|AB| = |CD| = 1$. Suppose that AB and CD intersect at a point O which is distinct from A and C, such that the angle $\angle AOC = 60°$. Show that $|AC| + |BD| \geq 1$.

Problem 5. A certain maths course recommends five textbooks, all costing a whole number of pounds and all at different prices.

Five keen students, Alex, Bobby, Chris, Daryl and Evelyn, each bought exactly two of the books. No two of them chose the same pair of books. Each of the five books were purchased by exactly two of the students. Alex spent a total of £30 on her books, Bobbie and Chris both spent £19, Daryl spent £17, and Evelyn spent £35.

What were the prices of the five books?

Problem 6. This problem has been edited out due to copyright reasons.

Problem 7. Does there exist a positive integer whose square has digit sum 2023?

Problem 8. Some elves and trolls – 60 of them in total – are seated at a table. Trolls always lie. Elves always speak the truth, unless they make a mistake, in which case they lie. Everybody claims to be sitting between an elf and a troll, but exactly two elves made a mistake! How many trolls are there at the table?

Problem 9. Let p be a prime number and let n be a positive integer. Suppose that $p - 1$ is a multiple of n, and that $n^3 - 1$ is a multiple of p. Prove that $4p - 3$ is a square number.

Problem 10. A finite number of smooth curves are drawn in the Euclidean plane $\mathbb{R}^2$ such that:

- each curve is either a loop or of infinite length,
- none of the curves are self-intersecting,
- the curves intersect at finitely many points, and
- each point of intersection lies on exactly two curves that meet tangentially at the point without crossing each other.

Let S be the union of these curves. Show that there exists a connected component of $\mathbb{R}^2 \setminus S$ whose boundary is smooth at all but 0 or 1 point.

Problem 11. Given a real number x, let $[x]$ denote the nearest integer to x, rounding up if x is halfway between integers. For example, $[1] = 1$, $[1.5] = 2$, and $[\pi] = 3$. Prove that for any natural number N, we have:

$$N = \sum_{n=1}^{\infty} \left[\frac{N}{2^n} \right].$$

Problem 12. Let $f : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function with the property that for all $n \in \mathbb{N}$, $f(f(n)) = 3n$. What is the value of $f(2023)$?

Problem 13. Let S be a set with a commutative and associative binary operation $*$. Suppose that for every x and y in S, there exists a z in S such that $x * z = y$. Prove that if a, b, $c \in S$ are such that $a * c = b * c$, then $a = b$.

Problem 14. This problem has been edited out due to copyright reasons.

Problem 15. Show that

$$\frac{\tan^{-1}(2)}{\pi}$$

is irrational.

Problem 16. Consider a 9×9 chessboard. Jordan places 65 ladybugs at the centres of distinct squares on the board in some fashion, such that each ladybug faces a non-diagonally adjacent square. The ladybugs all begin to crawl at a rate of 1 square per second. Once they reach the centre of a new square, they will turn left or right and continue their crawl. They are clever insects, so they choose left or right in such a way that they will never fall off of the edge of the board. Prove that eventually two ladybugs will meet in the centre of the same square.

2 Appendix and Clarifications

Problem 1.
The set of all allowed expressions is precisely defined as follows:

- E is an allowed expression if $E = n$ for any integer n, or if $E = a, b,$ or c.
- If E and F are allowed expressions, then the following are also allowed expressions:

$$E + F, \quad E - F, \quad EF, \quad \frac{E}{F}, \quad |E|.$$

Problem 7.
The digit sum of a number is the sum of its digits. For example, the digit sum of 1234 is $1 + 2 + 3 + 4 = 10$.

Problem 12.
A function $f : \mathbb{N} \to \mathbb{N}$ is said to be strictly increasing if for all $a, b \in \mathbb{N}$, $a < b$ implies $f(a) < f(b)$.

Problem 13.
A binary operation on a set S is a function $\star : S \times S \to S$. Instead of writing $\star(x, y)$, we write $x \star y$ for $x, y \in S$.

The operation $\star$ is called commutative if for all $x, y \in S$, $x \star y = y \star x$.

The operation $\star$ is called associative if for all $x, y, z \in S$, $x \star (y \star z) = (x \star y) \star z$.